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Effects of crack surface convection for rapid crack growth in a thermoelastic solid

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Abstract

As a Green's function for rapid steady-state crack growth with crack surface convection, semi-infinite Mode I crack growth at subcritical speeds in an unbounded solid under the action of compressive line loads moving on the crack surfaces is considered. A standard convection law that relates heat flux to change in temperature is employed, and the solid obeys the fully-coupled (dynamic) equations of thermoelasticity.

The use of robust asymptotic forms reduces the problem to the solution of coupled integral equations. These exhibit both Cauchy and Abel operators, but an exact solution is possible.

The solution indicates that convection can give rise to temperature changes in the crack plane that are both more prominent and extensive than those that occur for an insulated crack surface. Exact expressions for the thermoelastic Rayleigh speed, which is the critical crack speed, and for speeds that arise for a particular value of an important characteristic parameter are also presented. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Thermoelasticity; Convection; Rapid crack growth; Integral equations

1. Introduction

Studies (Brock, 1994, 1996a) of rapid crack growth based on the fully-coupled (dynamic) equations of thermoelasticity (Chadwick, 1960; Boley and Weiner, 1985) indicate the importance of thermoelastic effects at high crack speeds. However, the studies use a standard assumption that heat flux across the newly-created fracture surface is negligible, i.e. the crack surface is insulated. It is known (Ewalds and Wanhill, 1985) that crack surfaces may have fracture-altered granular make-ups. While this effective surface layer might be negligible in modeling elastic response, it could give rise to heat flux by

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convection. A standard (Boley and Weiner, 1985) layer convection model requires that the heat flux across any point on a surface be proportional to the temperature change at that point.

This article revisits, therefore, rapid crack growth in a thermoelastic solid by including this model. The same 2-D Mode I steady-state situation—a semi-infinite crack, driven by compressive loads moving on its surface, runs in an unbounded solid—treated by Brock (1994, 1996a) is again examined. However, it is now viewed as a Green's function for steady-state crack growth with convection. Therefore, crack edge plasticity is neglected. It will be seen that the fracture mechanics of the de facto brittle fracture problem is somewhat insensitive to thermal effects.

The problem is formulated in the next section, and a related problem that involves simpler boundary conditions is extracted. This related problem is then solved exactly in an integral transform space, and robust asymptotic forms used to reduce the original problem to a pair of coupled integral equations. They involve both Cauchy and Abel operators, yet an analytic solution is possible. The solution shows that convection makes crack plane temperature changes both more prominent and extensive than those that arise for an insulated crack surface.

2. Problem formulation

Consider an unbounded thermoelastic solid, with a crack of infinite width and semi-infinite length defined in terms of the Cartesian coordinates (x, y, z) as $(y = 0, x < 0)$. The cracked solid is at rest at a uniform (absolute) temperature T_0, when compressive line loads of magnitude P are applied to opposite faces of the crack, and moved in the positive x-direction with a constant subcritical speed v. This wedging action causes crack growth in the positive x-direction, and a steady-state is attained in which the crack speed is also v. This process is 2-D, so that z-dependence can be ignored, and Fig. 1 used as a schematic representation. There L is the crack edge-load separation distance and the xy-axes are fixed to the moving crack edge, i.e. $(y = 0, x < 0)$ always defines the crack.

Because this process is symmetric about the crack plane (x-axis), attention can be focused on the half-plane $y > 0$ by appending the relevant boundary conditions

$$y = 0, \quad x > 0\colon \quad \sigma_{xy} = u_y = \frac{\partial \theta}{\partial y} = 0 \tag{1a}$$

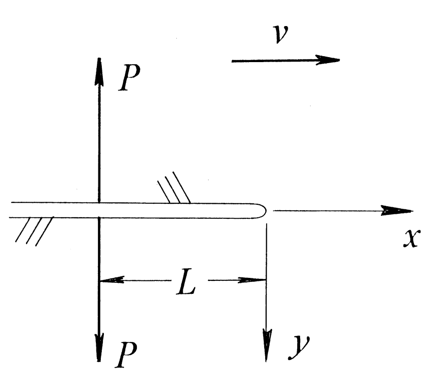

Fig. 1. Schematic of crack growing under moving compressive line loads.

$$y = 0, \quad x < 0: \quad \sigma_{xy} = \frac{\partial \theta}{\partial y} - \frac{\theta}{h_c} = 0, \quad \sigma_y = -P\delta(x + L) \tag{1b}$$

In (1) $\delta(\ )$ is the Dirac function, θ is the change in absolute temperature from T_0, $(u_x,\ u_y)$ are the only displacements, $(\sigma_{xy},\ \sigma_x,\ \sigma_y)$ are the relevant tractions, and $h_c > 0$ is a length that characterizes crack surface convection. If the convection represents a layer of effective thickness l on the crack surface, then the Biot number B_l for the crack would (Boley and Weiner, 1985) be

$$B_l = \frac{l}{h_c} \tag{2}$$

For this 2-D steady-state problem, field variables depend only on $(x,\ y)$, and time derivatives in the absolute (inertial) frame can be written as $-v\ \partial(\ )/\partial x$. Thus, from Chadwick (1960) and Brock (1996a) the governing equations of coupled thermoelasticity for $y > 0$ are

$$\left(\nabla^2 - m^2 c^2 \frac{\partial^2}{\partial x^2}\right)(u_x,\ u_y) + \left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}\right)[(m^2 - 1)\Delta + \chi\theta] = 0 \tag{3a}$$

$$h\nabla^2\theta + c\frac{\partial}{\partial x}\left(\theta - \frac{m^2\epsilon}{\chi}\Delta\right) = 0 \tag{3b}$$

$$\frac{1}{\mu}\sigma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \frac{1}{\mu}(\sigma_x,\sigma_y) = 2\left(\frac{\partial u_x}{\partial x},\frac{\partial u_y}{\partial y}\right) + (m^2 - 2)\Delta + \chi\theta \tag{3c}$$

In (3) $(\nabla^2,\ \Delta)$ are the 2-D Laplacian and dilatation, and

$$\chi = \chi_0(4 - 3m^2), \quad \epsilon = \frac{T_0}{c_v}\left(\frac{\chi v_r}{m}\right)^2, \quad h = \frac{k v_r}{\mu m c_v}, \quad m = \frac{v_d}{v_r}, \quad c = \frac{v}{v_d} \tag{4}$$

where $(\chi_0,\ c_v,\ k,\ \mu)$ are, respectively, the thermal expansion coefficient, specific heat, thermal conductivity and shear modulus. The parameters $(v_r,\ v_d)$ are the rotational and isothermal dilatational wave speeds, while $(\epsilon,\ h)$ are the dimensionless thermoelastic coupling constant and thermoelastic characteristic length. It can be shown (Chadwick, 1960; Brock, 1992) that for many materials

$$\epsilon \approx O(10^{-2}), \quad h \approx O(10^{-4})\ \mu\text{m}, \quad m > \sqrt{2} \tag{5}$$

In addition, we expect $(\sigma_{xy},\ \sigma_x,\ \sigma_y,\ \theta)$ to vanish as $\sqrt{x^2 + y^2} \to \infty$, $y \geq 0$, and for these fields to be non-singular everywhere except perhaps at $y = 0$, $x = 0$ and $y = 0$, $x = -L$. At this point, we define subcritical crack-load speed to be that which does not exceed v_r, i.e. $0 < c < 1/m$.

3. Related problem and asymptotic solution

To address this problem, we consider first the related problem with unmixed conditions

$$y = 0: \quad \sigma_{xy} = 0, \quad u_y = U(x)H(-x), \quad \frac{\partial \theta}{\partial y} = G(x)H(-x) \tag{6}$$

Here $H(\ )$ is the Heaviside function, while $U(x)$ and $G(x)$ are unknown functions which vanish identically for $x > 0$, are no worse than integrally singular for $x < 0$ and which remain finite as $x \to -\infty$. Eqs. (3)–(5) and the boundedness/singularity conditions imposed on the original problem hold as well. In view of (1), the related problem will yield the solution for the original problem if (U, G) are such that

$$y = 0, \quad x < 0: \quad \sigma_y = -P\delta(x + L), \quad \theta = h_c G(x) \tag{7}$$

The unknown functions $(2U, h_c G)$ can now be interpreted as the crack-opening displacement and the crack surface temperature change. Thus, the condition

$$U(0-) = 0 \tag{8}$$

must also hold.

To consider the related problem, the bilateral Laplace transform/inversion operator pair (van der Pol and Bremmer, 1950)

$$g^* = \int_{-\infty}^{\infty} g(x)\, \mathrm{e}^{-px}\, \mathrm{d}x,$$

$$g(x) = \frac{1}{2\pi i}\int g^*\, \mathrm{e}^{px}\, \mathrm{d}p \tag{9a,b}$$

is introduced, where p is generally complex and integration in (9b) is along the Bromwich contour. Application of (9a) to (3) in view of the boundedness conditions leads to the relevant transform set

$$\begin{bmatrix} u_x^* \\ \dfrac{1}{p^2}\theta^* \\ \dfrac{1}{\mu p}\sigma_y^* \end{bmatrix} = \begin{bmatrix} -p & -p & 1 \\ \omega_+ & \omega_- & 0 \\ -Kp & -Kp & -2 \end{bmatrix} \begin{bmatrix} A_+\, \mathrm{e}^{-\alpha_+ y} \\ A_-\, \mathrm{e}^{-\alpha_- y} \\ B\, \mathrm{e}^{-\beta y} \end{bmatrix} \tag{10a}$$

$$\begin{bmatrix} u_y^* \\ \dfrac{1}{p^2}\dfrac{\partial\theta^*}{\partial y} \\ \dfrac{1}{\mu p}\sigma_{xy}^* \end{bmatrix} = \begin{bmatrix} -1 & -1 & -p \\ \omega_+ & \omega_- & 0 \\ -2 & -2 & Kp \end{bmatrix} \begin{bmatrix} -\alpha_+ A_+\, \mathrm{e}^{-\alpha_+ y} \\ -\alpha_- A_-\, \mathrm{e}^{-\alpha_- y} \\ -\dfrac{1}{\beta} B\, \mathrm{e}^{-\beta y} \end{bmatrix} \tag{10b}$$

for $y > 0$. Here the coefficients $(A_\pm, B)$ are as-yet-undetermined functions of p and

$$\alpha_\pm = a_\pm \sqrt{p}\sqrt{-p}, \quad \beta = b\sqrt{p}\sqrt{-p}, \quad \omega_\pm = \frac{m^2}{\chi}(1 - c^2 - a_\pm^2) \tag{11a}$$

$$a_\pm = \sqrt{1 + \frac{c}{p}(\tau_+ \pm \tau_-)^2}, \quad b = \sqrt{1 - m^2 c^2}, \quad K = m^2 c^2 - 2 \tag{11b}$$

$$2\tau_{\pm}\sqrt{\left(\sqrt{-cp}\pm\frac{1}{\sqrt{h}}\right)^2+\frac{\epsilon}{h}}, \quad \omega_+\omega_- = \frac{m^4c^3\epsilon}{\chi^2 hp} \tag{11c}$$

where branch cuts must be chosen so that $\mathrm{Re}(\alpha_\pm, \beta) \geq 0$ in the cut p-plane. Operation on (6) with (9a) and use of (10) then gives the equations necessary to determine $(A_\pm, B)$ as

$$A_\pm = \frac{\pm 1}{\alpha_\pm(\omega_- - \omega_+)}\left(\frac{K\omega_\mp}{m^2c^2}\int_{-\infty}^{0} U\,\mathrm{e}^{-pt}\,\mathrm{d}t \pm \frac{1}{p^2}\int_{-\infty}^{0} G\,\mathrm{e}^{-pt}\,\mathrm{d}t\right) \tag{12a}$$

$$B = \frac{2\beta}{m^2c^2p}\int_{-\infty}^{0} U\,\mathrm{e}^{-pt}\,\mathrm{d}t \tag{12b}$$

With (10) and (12) available, the related problem is essentially solved. Solution of the original problem requires in view of (7) that expressions for (σ_y^*, θ^*) be inverted for $(y = 0, x < 0)$ by means of (9b). Such an inversion process, however, gives expressions for (σ_y, θ) that lead to a semi-numerical process for obtaining the unknown functions (U, G). We follow, therefore, Brock (1996b) and Brock and Georgiadis (1997) and make use of asymptotic results: bilateral Laplace transforms valid for small $|hp|$ give inversions that are valid for large $|x/h|$ (van der Pol and Bremmer, 1950); because h is defined by (4) and (5), the inversions will be robust.

Eqs. (11) and (12) are, therefore, substituted into (10), and the results expanded in Taylor series for $|hp| \ll 1$. Keeping the lowest-order terms then gives, for example, the asymptotic form

$$\theta^* = \frac{-K\epsilon}{\chi(1+\epsilon)a}\frac{\sqrt{p}}{\sqrt{-p}}\mathrm{e}^{-ay\sqrt{p}\sqrt{-p}}\int_{-\infty}^{0}\frac{\mathrm{d}U}{\mathrm{d}x}\mathrm{e}^{-pt}\,\mathrm{d}t - \frac{\sqrt{h}}{\sqrt{c(1+\epsilon)}\sqrt{-p}}\mathrm{e}^{-y\sqrt{c/h(1+\epsilon)}\sqrt{-p}}\int_{-\infty}^{0} G\,\mathrm{e}^{-pt}\,\mathrm{d}t \tag{13}$$

for $y > 0$. In (13) the positive real quantity

$$a = \sqrt{1 - \frac{c^2}{1+\epsilon}} \tag{14}$$

is a manifestation of the thermoelastic (adiabatic) dilatational wave speed $v_\mathrm{d}\sqrt{1+\epsilon}$. Because (a, b) are real and positive, the conditions on $(\alpha_\pm, \beta)$ in the cut p-plane lead to the branch cuts $\mathrm{Im}(p)=0$, $\mathrm{Re}(p) < 0$ and $\mathrm{Im}(p)=0$, $\mathrm{Re}(p) > 0$ for the radicals $(\sqrt{p}, \sqrt{-p})$, respectively. In developing (13), Eqs. (8) and (9) and the fact that $U = 0$ for $x > 0$ led to the substitution

$$pU^* = \left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^* \tag{15}$$

and, thence, to the appearance of the gradient $\mathrm{d}U/\mathrm{d}x$ as the unknown function. This is of no consequence, since the steady-state nature of the process limit solution determination to within an arbitrary rigid body motion.

Assuming that the t-integration and inversion process can be interchanged, (13) produces the two transform types

$$\frac{\sqrt{p}}{\sqrt{-p}}\mathrm{e}^{-pt-ay\sqrt{p}\sqrt{-p}}, \quad \frac{1}{\sqrt{-p}}\mathrm{e}^{-pt-y\sqrt{c/h(1+\epsilon)}\sqrt{-p}} \tag{16a,b}$$

for $y > 0$, $t < 0$. Substitution of type (16a) into (9b) gives an integration for which the Bromwich contour can be taken as the entire Im(p)-axis. For $x > t$ and $x < t$, respectively, Cauchy theory allows this contour to be switched to the paths around the branch cuts Im(p) = 0, Re(p) < 0 and Im(p) = 0, Re(p) > 0. Similarly, substitution of type (16b) into (9b) leads to an integration around the branch cut Im(p) = 0, Re(p) < 0 for $x < t$. The integrations in each case can be found in standard tables (Gradshteyn and Ryzhik, 1980), with the result that (13) yields the formula

$$\theta = \frac{-K\epsilon}{\chi(1+\epsilon)a}\frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{d}U}{\mathrm{d}x}\frac{t-x}{(t-x)^2+a^2y^2}\,\mathrm{d}t - \frac{\sqrt{h}}{\sqrt{\pi c(1+\epsilon)}}\int_{x}^{0} G\mathrm{e}^{-\dfrac{c(1+\epsilon)y^2}{4h(t-x)}}\frac{\mathrm{d}t}{\sqrt{t-x}} \tag{17}$$

for $y > 0$, $|x/h| \gg 1$.

4. Original problem: integral equations

In view of (17) and the corresponding result for σ_y, the original problem can now be reduced in light of (8) to the coupled integral equations

$$\frac{R}{m^2c^2a}\frac{1}{\pi}⨍_{-\infty}^{0}\frac{\mathrm{d}U}{\mathrm{d}x}\frac{\mathrm{d}t}{t-x} + \frac{K\chi h}{m^2ca(1+\epsilon)}\frac{1}{\pi}⨍_{-\infty}^{0} G\frac{\mathrm{d}t}{t-x} = -\frac{P}{\mu}\delta(x+L) \tag{18a}$$

$$\frac{K\epsilon}{\chi(1+\epsilon)a}\frac{1}{\pi}⨍_{-\infty}^{0}\frac{\mathrm{d}U}{\mathrm{d}x}\frac{\mathrm{d}t}{t-x} + \frac{\sqrt{h}}{\sqrt{\pi c(1+\epsilon)}}\int_{x}^{0}\frac{G\,\mathrm{d}t}{\sqrt{t-x}} + h_{\mathrm{c}}G = 0 \tag{18b}$$

for $x < 0$. Here $⨍$ denotes Cauchy principal value integration and

$$R = 4ab - K^2 \tag{19}$$

is the asymptotic thermoelastic Rayleigh function in terms of the dimensionless crack speed c. The quantity $R = R(c)$ exhibits the roots $c = \pm(0, c_R)$, $0 < c_R < 1/m$ in the cut c-plane, and the value of c_R, which is the asymptotic thermoelastic Rayleigh speed non-dimensionalized with respect to v_{d}, can be found by standard (Kunz, 1957) numerical root-finding schemes. Either as a check or an alternative, one can use an analytical approach (Brock, 1997, 1998) to obtain the exact formula

$$c_R = \sqrt{2\left(m^2 - \frac{1}{1+\epsilon}\right)\frac{1}{m^2F_0}}, \quad \ln\ F_0 = \frac{1}{\pi}\int_{1/m}^{\sqrt{1+\epsilon}}\frac{\Phi\,\mathrm{d}t}{t} \tag{20a}$$

$$\Phi = \tan^{-1}\frac{4\sqrt{1+\epsilon-t^2}\sqrt{m^2t^2-1}}{\sqrt{1+\epsilon}(m^2t^2-2)^2} \tag{20b}$$

Because the nature of (18a) changes when R vanishes, we modify the definition of subcritical crack growth to be

$$0 < c < c_R \tag{21}$$

Linearly combining (18a,b) produces the partly-coupled set

$$\frac{1}{\pi}\int_{-\infty}^{0}\left[\frac{\mathrm{d}U}{\mathrm{d}x}+\frac{Kc\chi h}{(1+\epsilon)R}G\right]\frac{\mathrm{d}t}{t-x}=-\frac{m^2c^2a}{R}\,\frac{P}{\mu}\delta(x+L) \tag{22a}$$

$$\frac{1}{\pi}\int_{-\infty}^{0}\frac{G\,\mathrm{d}t}{t-x}-\frac{Ra}{K^2\epsilon\sqrt{h}}\left(\frac{1+\epsilon}{c}\right)^{3/2}\int_{x}^{0}\frac{G\,\mathrm{d}t}{\sqrt{\pi(t-x)}}-\frac{h_{\mathrm{c}}\;Ra(1+\epsilon)^2}{h\epsilon K^2c}G=-\frac{m^2ca(1+\epsilon)}{K\chi h}\,\frac{P}{\mu}\delta(x+L) \tag{22b}$$

for $x<0$. Eq. (22a) is an inhomogeneous Cauchy singular integral equation for a linear combination of $(\mathrm{d}U/\mathrm{d}x,\,G)$. Its solution is readily found by standard techniques (Carrier et al., 1966) as

$$\frac{\mathrm{d}U}{\mathrm{d}x}+\frac{Kc\chi h}{(1+\epsilon)R}G=-\frac{m^2c^2a}{\pi R}\,\frac{P}{\mu}\,\frac{\sqrt{L}}{\sqrt{-x}(x+L)} \tag{23}$$

To address (22b) it is convenient to introduce the variable $x=-\xi$, so that we have

$$\frac{1}{\pi}\int_{0}^{\infty}\frac{G\,\mathrm{d}\tau}{\tau-\xi}+\frac{1}{\sqrt{\mathrm{d}}}\int_{0}^{\xi}\frac{G\,\mathrm{d}\tau}{\sqrt{\pi(\xi-\tau)}}+AG=G_0\delta(\xi-L)(\xi>0) \tag{24}$$

where

$$G_0=\frac{m^2ca(1+\epsilon)}{K\chi h}\,\frac{P}{\mu},\quad A=\frac{Ra}{cK^2}\,\frac{(1+\epsilon)^2}{\lambda\epsilon},\quad \lambda=\frac{h}{h_{\mathrm{c}}},\quad \frac{d}{h}=\left(\frac{cK^2}{Ra}\right)^2\frac{\epsilon^2c}{(1+\epsilon)^3} \tag{25}$$

In (25) the positive real parameters $(A,\,d)$ are, respectively, a dimensionless constant and a characteristic length. Both are functions of crack speed (c) and thermoelastic properties $(m,\,\epsilon,\,h)$, but A also depends on convection through the dimensionless convection parameter λ.

5. Original problem: completion of solution

Eq. (24) exhibits both Cauchy and Abel operators, but can be addressed by introducing the unilateral Laplace transform/inversion operator pair (Sneddon, 1972)

$$\hat{g}=\int_{0}^{\infty}g(\xi)\,\mathrm{e}^{-s\xi}\,\mathrm{d}\xi,\quad g(\xi)=\frac{1}{2\pi i}\int\hat{g}\,\mathrm{e}^{s\xi}\,\mathrm{d}s \tag{26a,b}$$

Here $\mathrm{Re}(s)>0$ and is large enough to ensure existence of (26a), and integration in (26b) is along a Bromwich contour. Application of (26a) to (24) gives the integral equation

$$\frac{1}{\pi}\int_{0}^{\infty}\frac{\hat{G}\,\mathrm{d}u}{s-u}+\left(\frac{1}{\sqrt{sd}}+A\right)\hat{G}=G_0\,\mathrm{e}^{-sL}(\mathrm{Re}(s)>0) \tag{27}$$

for the transform $\hat{G}$. All but the first term in (27) follow from their counterparts in (24) through use of standard tables (Abramowitz and Stegun, 1970) and the convolution theorem. The first term follows from its counterpart in (24) by assuming that integralibility of $G(\xi)$, $\xi>0$ is sufficient to allow the orders of Cauchy principal value and transform integrations to be interchanged.

The relation (27) is an inhomogeneous Cauchy singular integral equation with a variable coefficient. Its solution is found by standard techniques (Carrier et al., 1966) as

$$\hat{G} = \frac{C}{s}\frac{(sd)^{\alpha}\,\mathrm{e}^{-\omega}}{\sqrt{sd+(1+A\sqrt{sd})^2}} + \frac{G_0\sqrt{sd}}{\sqrt{sd+(1+A\sqrt{sd})^2}}\left[\frac{(1+A\sqrt{sd})\,\mathrm{e}^{-sL}}{\sqrt{sd+(1+A\sqrt{sd})^2}}\right.$$

$$\left. + (sd)^{\alpha}\frac{\mathrm{e}^{\omega}}{\pi}\int_0^{\infty}\frac{\sqrt{ud}\,\mathrm{e}^{-\omega-uL}}{(ud)^{\alpha}(u-s)}\frac{\mathrm{d}u}{\sqrt{ud+(1+A\sqrt{ud})^2}}\right] \tag{28}$$

where s can be treated as positive real, C is an arbitrary constant and

$$\alpha = \frac{1}{\pi}\tan^{-1}A\left(0\le\alpha\le\frac{1}{2};A\ge 0\right),\quad \omega = \frac{1}{\pi}\int_0^{\infty}\tan^{-1}\frac{1}{A+(1+A^2)\sqrt{t}}\frac{\mathrm{d}t}{t-sd} \tag{29}$$

In (28), $\sqrt{s}$ has a branch cut $\mathrm{Im}(s)=0$, $\mathrm{Re}(s) < 0$ in order that its real part be positive indefinite in the cut s-plane. Boundedness of G as $\xi\to\infty$ and the Tauberian theorems require that $s\hat{G}$ be finite as $s\to 0$. In this light, behavior of (28) demonstrates that

$$C = 0 \tag{30}$$

Inversion of (28) and (30) is accomplished by direct use of (26b), with the entire $\mathrm{Im}(s)$-axis serving as the Bromwich contour. However, because $\mathrm{Re}(\sqrt{s})\ge 0$, (28) exhibits no poles or zeros in the cut s-plane and a branch cut only along the negative $\mathrm{Re}(s)$-axis. Therefore, Cauchy theory can be used to transform the integration path to a contour surrounding the cut. Upon then introducing the robust approximation

$$\omega = \left(\alpha-\frac{1}{2}\right)\ln(sd)+\omega_0,\quad \omega_0 = \frac{2}{\pi}\int_0^{\infty}\frac{\ln\,t}{t^2+(1+At)^2}\,\mathrm{d}t \tag{31}$$

valid for $|hs|\ll 1(\xi/h\gg 1)$ and the original variable x, it can be shown that

$$h_cG = \frac{m^2ca(1+\epsilon)}{K\chi\lambda}\frac{P}{\mu d}\frac{H(-x-L)}{\pi\sqrt{-x-L}}\int_0^{\infty}\frac{[x+L+(1-A^2)t]\sqrt{t}\,\mathrm{e}^{-t/d}\,\mathrm{d}t}{[x+L+(1-A^2)t]^2+4A^2t^2}$$

$$+\frac{m^2ca(1+\epsilon)}{K\chi\lambda}\frac{P}{\mu d}\frac{H(-x)}{\pi^2}\int_0^{\infty}\frac{\sqrt{t}\,\mathrm{e}^{-t/d}\,\mathrm{d}t}{\sqrt{\rho(-x,\,t)}}\int_0^{t/d}\frac{\sqrt{u}\,\mathrm{e}^{u/d}}{\sqrt{\rho(L,\,u)}}\frac{\mathrm{d}u}{xu+Lt}\sin[\psi(-x,\,t)+\psi(L,\,u)] \tag{32a}$$

$$\rho(\alpha,\,\beta) = \sqrt{[\alpha-(1+A^2)\beta]^2+4A^2\alpha\beta},\quad 2\psi(\alpha,\,\beta) = \tan^{-1}\frac{2A\sqrt{\alpha\beta}}{\alpha-(1+A^2)\beta} \tag{32b}$$

With (23) and (32) available, the original problem is solved. In view of (1b) and (6), (32a) is also the asymptotic temperature change θ on the crack surface. From (17) the asymptotic temperature change on the plane $y = 0$ ahead $(x/h\gg 1)$ of the crack edge is

$$\theta = \frac{-K\epsilon}{\chi(1+\epsilon)}\frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{d}U}{\mathrm{d}x}\frac{\mathrm{d}t}{t-x} \tag{33}$$

Substitution of (23) and (32) and the use of Cauchy residue theory produces a result whose dominant contribution can be written as

$$\theta = -\frac{Km^2c^2a\epsilon}{\chi(1+\epsilon)R}\,\frac{P}{\mu}\,\frac{\sqrt{L}}{\sqrt{x}(x+L)} + \frac{Km^2c^2a\epsilon}{\chi(1+\epsilon)R}\,\frac{P}{\mu}\,\frac{A}{2\pi(1+A^2)}\,\frac{1}{x+L}$$
$$+\frac{Km^2c^2a\epsilon}{\chi(1+\epsilon)R}\,\frac{P}{\mu}\,\frac{1}{\pi^2 x}\int_0^\infty \frac{\sqrt{t}\,\mathrm{d}t}{\sqrt{\rho(L,\,t)}}\ \cos\ \psi(L,\,t)\int_0^\infty \frac{\mathrm{d}u}{(ut+Ld)\sqrt{1+(A+\sqrt{u})^2}} \tag{34}$$

for $y = 0,\ x/h \gg 1$.

For large $|x+L|$ Eq. (32) behaves as

$$h_c G \approx -\frac{m^2ca(1+\epsilon)}{K\chi\lambda}\,\frac{P}{\mu}\sqrt{\frac{d}{\pi}}\frac{1}{2(-x-L)^{3/2}} - \frac{m^2ca(1+\epsilon)}{K\chi\lambda}\,\frac{P}{\mu}\,\frac{1}{\pi^2(-x)^{3/2}}\int_0^\infty$$
$$\times\left[1+\frac{1}{2}\sqrt{\frac{\pi d}{t}}\ \mathrm{e}^{t/d}\ \mathrm{erfc}(\sqrt{t/d})\right]\frac{\mathrm{d}t}{\sqrt{\rho(L,\,t)}}\ \sin\ \psi(L,\,t) \tag{35}$$

In view of (23) this implies that the contribution of G to the crack opening displacement will behave as $O(|x+L|^{-1/2})$ for large $|x+L|$, thereby satisfying solution boundedness requirements.

6. Behavior of the parameter *A*

The problem solution is clearly sensitive to the positive dimensionless parameter A defined in (25). We now, therefore, examine its behavior: in regard to its variation with crack speed (c), (25) demonstrates that $\mathrm{d}^2A/\mathrm{d}c^2 \le 0,\ 0 < c < c_R$ and that

$$A \approx \left(m^2 - \frac{1}{1+\epsilon}\right)\frac{(1+\epsilon)^2}{2\lambda\epsilon}c, \quad \frac{d}{h} \approx \frac{\epsilon^2}{4(1+\epsilon)^2}\,\frac{1}{\left(m^2-\dfrac{1}{1+\epsilon}\right)^2 c}(c \approx 0) \tag{36a}$$

$$A \approx \frac{2m^4c_R^2}{K_R^2}\,\frac{(1+\epsilon)^2}{\lambda\epsilon}a_R F_R(c_R - c), \quad \frac{d}{h} \approx \left(\frac{K_R^2}{2m^4c_R^2 a_R F_R}\right)^2\left(\frac{\epsilon}{1+\epsilon}\right)^2\frac{c_R}{(c_R-c)^2}(c \approx c_R) \tag{36b}$$

where

$$\ln\ F_R = \frac{2}{\pi}\int_{1/m}^{\sqrt{1+\epsilon}} \frac{t\Phi\,\mathrm{d}t}{t^2 - c_R^2}, \quad a_R = \sqrt{1-\frac{c_R^2}{1+\epsilon}}, \quad K_R = m^2c_R^2 - 2 \tag{37}$$

and $(c_R,\ \Phi)$ are given in (20). This behavior suggests the schematic representation of A as a function of c in Fig. 2. The principle of the argument (Hille, 1959) shows that the slope quantity $\mathrm{d}A/\mathrm{d}c$ exhibits the real roots $c = \pm c_0,\ 0 < c_0 < c_R$ in the cut c-plane. The value of c_0 can be obtained numerically and, as with c_R, an analytical procedure (Brock, 1997, 1998) gives, either as a check or an alternative,

$$c_0 = \frac{\sqrt{2\left(m^2 - \frac{1}{1+\epsilon}\right)}}{\sqrt{1 + \left(\frac{2}{m\sqrt{1+\epsilon}}\right)^3}} \frac{1}{m^2 S_0}, \quad \ln\ S_0 = \frac{1}{\pi}\int_{1/m}^{\sqrt{1+\epsilon}} \frac{\Psi\ \mathrm{d}t}{t} \tag{38a}$$

$$\Psi = \tan^{-1}\ \tan\ \Phi\left[1 + \frac{4t(1+\epsilon-t^2)}{(1+\epsilon)(m^2t^2-1)} + \frac{t^2}{1+\epsilon}\frac{2m^2t^2 - 1 - m^2(1+\epsilon)}{m^2t^2-1}\right] \tag{38b}$$

where Φ is defined by (20b). For some insight into determining the crack speeds possible for a given value of A, consider the case $A = 1$. Fig. 2 shows that when $A(c_0) > 1$, there will be two such values (c_1, c_2), where $0 < c_1 < c_0 < c_2 < c_R$. The values can be found numerically as roots of the quantity $1-A(c)$, $A(c_0) > 1$. To provide a check on these values, the principle of the argument can be used to show that $1-A(c)$ has three real roots (c_1, c_2, $-c_3$), $c_1 < c_2 < c_R < c_3 < 1/m$ in the cut c-plane when $A(c_0) > 1$. Then, the analytical procedure used for (20) and (38) gives the three equations

$$c_1^2 + c_2^2 + c_3^2 = \mathrm{I}, \quad c_1^2c_2^2 + c_2^2c_3^2 + c_3^2c_1^2 = \mathrm{II}, \quad c_1^2c_2^2c_3^2 = \mathrm{III} \tag{39}$$

where the positive real quantities (I, II, III) are defined by

$$\mathrm{I} = c_0^2 + c_R^2 + \frac{(\lambda\epsilon)^2}{(\lambda\epsilon)^2 + (1+\epsilon)^3}\left[\frac{C_0}{c_0^2c_R^2} + \frac{1}{c_R^2 - c_0^2}\left(\frac{B_0}{c_0^2} + \frac{B_R}{c_R^2}\right)\right] \tag{40a}$$

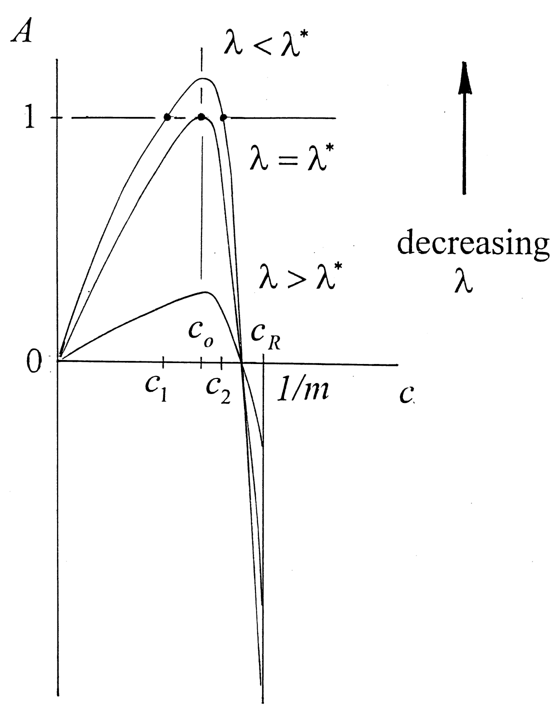

Fig. 2. Schematic of dimensionless parameter A as function of non-dimensionalized crack speed c.

$$\mathrm{II} = c_0^2 c_R^2 + \frac{(\lambda\epsilon)^2}{(\lambda\epsilon)^2 + (1+\epsilon)^3}\left[\left(\frac{1}{c_0^2} + \frac{1}{c_R^2}\right)C_0 + \frac{1}{c_R^2 - c_0^2}\left(\frac{c_R^2}{c_0^2}B_0 + \frac{c_0^2}{c_R^2}B_R\right)\right] \tag{40b}$$

$$\mathrm{III} = C_0 \tag{40c}$$

In (40) the positive real quantities

$$B_0 = \frac{K_0^3}{m^6}\frac{1 - A(c_0)}{D_0}, \quad B_R = \frac{-K_R^3}{m^6 D_R}, \quad C_0 = \frac{4}{m^6 E_0} \tag{41}$$

appear, where

$$\ln(E_0, D_0, D_R) = \frac{2}{\pi}\int_{1/m}^{\sqrt{1+\epsilon}} \Omega\left(\frac{1}{t}, \frac{t}{t^2 - c_0^2}, \frac{t}{t^2 - c_R^2}\right) \mathrm{d}t \tag{42a}$$

$$\Omega = \tan^{-1}\frac{2(1+\epsilon)(1+\epsilon-t^2)\tan\Phi}{(1+\epsilon)(1+\epsilon-t^2)(1+\tan^2\Phi) - \left(\frac{\lambda\epsilon t}{1+\epsilon}\right)^2}, \quad K_0 = m^2 c_0^2 - 2 \tag{42b}$$

and (c_R, Φ), K_R and c_0 are defined by (20), (37) and (38). Eqs. (39) combine to give the cubic equation

$$\zeta^3 - \mathrm{I}\zeta^2 + \mathrm{II}\zeta - \mathrm{III} = 0 \tag{43}$$

whose three positive real roots $\zeta = (c_1^2, c_2^2, c_3^2)$ can be obtained from standard (Abramowitz and Stegun, 1970) formulas.

Eqs. (39) are valid only for $A(c_0) > 1$, but in the limit as $A(c_0) = 1$ give the appropriate result that $c_1 = c_2 = c_0$. For illustration, let (λ^*, h_c^*) denote the values for which this case, i.e. $(1-A = \mathrm{d}A/\mathrm{d}c = 0)$, occurs. Then, for the generic steel material with properties

$$m \approx \sqrt{3}, \quad \epsilon \approx 0.01, \quad h \approx 1.67(10^{-4})\ \mu\mathrm{m} \tag{44}$$

(20) and (38) give

$$c_0 \approx 0.4, \quad \lambda^* \approx 34, \quad h_c^* \approx 4.91(10^{-6})\ \mu\mathrm{m} \tag{45}$$

To put this value of h_c^* in perspective, consider that for an effective crack surface layer thickness of $O(10^{-4}$–$10^{-3})$ μm, i.e. on the scale of crystal lattice dimensions (Guy, 1960), (2) gives a Biot number $B_l = B_l^*$ of $O(10$–$10^2)$.

7. Insulated limit case

The heat flux condition (1b) encapsulates the two limit cases of an insulated crack surface ($h_c \to \infty$) and a crack surface which allows no temperature change ($h_c \to 0$). In the former case, $A \to \infty$, the relation (34) for the temperature change ahead of the crack reduces to

$$\theta = -\frac{Km^2c^2 a\epsilon}{\chi(1+\epsilon)R}\frac{P}{\pi\mu}\frac{\sqrt{L}}{\sqrt{x}(x+L)} \quad (y = 0,\ x/h \gg 1) \tag{46}$$

In examining (32) it can be shown that the function G itself vanishes for all $0 < c < c_R$ when $h_c \to \infty$, but that the crack surface temperature change $h_c G$ itself vanishes everywhere except at $x = -L$. That is,

$$h_c G \to \frac{Km^2c^2\epsilon}{\chi(1+\epsilon)R}\,\frac{P}{\mu}\delta(x+L) \quad (y=0,\ x<0,\ |x/h| \gg 1) \tag{47}$$

This result is confirmed by a direct inversion of (28) and (30), and is identical to the steady-state temperature change on a thermoelastic half-space due to a moving line load (Brock and Georgiadis, 1997). Because $\chi < 0$ and $K < 0$, $0 < c < c_R$, (46) and (47) show that the asymptotic temperature of an insulated crack surface increases at the line load, while it decreases ahead of such a crack.

In regard to steady-state crack speed, use of (36) in (23), (32) and (34) shows that solution response is generally unbounded in the limit as $c \to c_R$. In the insulated limit $h_c \to \infty$, the case $c \to c_R$ remains critical, (46) and (47) behave, respectively, as

$$\theta = \frac{m^2\epsilon}{\chi[m^2(1+\epsilon)-1]}\,\frac{P}{\pi\mu}\,\frac{\sqrt{L}}{\sqrt{x}(x+L)} \quad (y=0,\ x/h \gg 1) \tag{48a}$$

$$h_c G \to \frac{-2m^2\epsilon}{\chi[m^2(1+\epsilon)-1]}\,\frac{P}{\mu}\delta(x+L) \quad (y=0,\ x<0,\ |x/h| \gg 1) \tag{48b}$$

These results constitute, of course, the results for an equilibrium crack.

8. Convection effects for the general case

Examination of (32) and (47) shows that, in the general case (finite h_c), convection alters the insulated limit property that the steady-state asymptotic crack surface temperature change occurs essentially only at the moving load. This phenomenon that convection renders steady-state temperature changes more prominent at distances from the load is also apparent ahead of the crack: the second and third terms in (34) behave for large $x+L$ as $O(1/x)$, while the first term, which corresponds to the insulated limit case (46), behaves as $O(1/x^{3/2})$. By the same process that led to (18a), it can be shown that

$$\frac{1}{\mu}\sigma_y = \frac{R}{m^2c^2a}\,\frac{1}{\pi}\int_{-\infty}^{0}\left[\frac{\mathrm{d}U}{\mathrm{d}x} + \frac{Kc\chi h}{(1+\epsilon)R}G\right]\frac{\mathrm{d}t}{t-x}(y=0,\ x/h \gg 1) \tag{49}$$

Substitution of (23) and use of Cauchy residue theory reduces (49) to

$$\frac{1}{\mu}\sigma_y = \frac{P}{\pi\mu}\,\frac{\sqrt{L}}{\sqrt{x}(x+L)} \quad (y=0,\ x/h \gg 1) \tag{50}$$

This result indicates that the steady-state asymptotic crack-opening (normal) stress ahead of the crack is unaffected by convection. As (23) and (32) show, this is in contrast to the crack-opening displacement. Thus, brittle fracture mechanics based on traction-displacement gradient behavior at the crack edge (Ewalds and Wanhill, 1985) would suggest that this de facto brittle crack problem is insensitive to thermal effects. However, as noted at the outset, the model treated here is to be viewed as a Green's function for steady-state crack growth with convection.

9. Brief summary

Crack surface convection in a Green's function for 2-D steady-state Mode I crack growth at a constant subcritical speed in a fully-coupled thermoelastic solid was examined here. The possibility of convection was based on the recognition that crack surfaces may exhibit effective layers of fracture-altered material, and a standard model which enforces proportionality between the crack surface heat flux and its change in temperature was employed.

The small thermoelastic characteristic length allowed the problem to be recast in terms of robust asymptotic functions that reduced it to the solution of coupled integral equations. The equations exhibited both Cauchy and Abel operators, but could be solved exactly. The solutions showed that convection gave temperature change fields that, as least in the crack plane, were both more prominent and extensive than those generated for an insulated surface. Besides the exact asymptotic solution, exact formulas for some crack speeds, including critical speed itself, i.e. the thermoelastic Rayleigh speed, were given.

This article, in summary, extended earlier work for standard (insulated surface) steady-state crack growth, which found that thermoelastic coupling effects can be important at high crack speeds. The present results suggest, therefore, that crack surface convection renders these effects even more prominent. It should also be noted that work is also underway which applies these results to rapid steady-state quasi-brittle crack growth, i.e. crack edge plastic effects are included.

In closing, a few observations are in order: first, the exact formulas for the speeds mentioned above were intended as checks on and alternatives to straight numerical determination processes. It is recognized that use of the formulas for calculation would not necessarily be more efficient. However, their existence would make possible future studies of speed variation with material parameters and crack speed more tractable.

Then, this article treated the convection-producing layer on the crack surface to be of a uniform thickness consistent with the idea that the layer is due to fracture-induced alterations of granular make-up at the surface itself. It is, however, known (Ewalds and Wanhill, 1985) that fracture surface profiles are, on the small-scale, irregular and transient studies of insulated half-space (Brock et al., 1996) show that surface thermal response can be influenced by even small-scale non-planarity. Therefore, future work on crack-surface convection will consider variable layer thicknesses.

Finally, it is noted that a key mathematical operation in the present analysis reduced an integral equation with Cauchy and Abel operators by unilateral Laplace transforms to one of a standard Cauchy type. The original equation followed from the inversion of bilateral Laplace transforms, which suggests that a corresponding equation in that transform space could have been formulated and addressed by a standard (Stakgold, 1971) Wiener–Hopf technique. However, that approach requires in fact complicated product- and sum-splitting operations which, moreover, lead to bilateral transforms whose inversions require much effort to produce expressions as tractable as those developed here.

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